Tutorial 8: Selected problems of Assignment 7

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Q1) (HW 7, Q6) Let (X,d) and (Y,ρ) be metric spaces such that (Y,ρ) is complete. Show that every uniformly continuous map $f: E \subseteq X \to Y$ ($E \neq \emptyset$) extends to a uniformly continuous map $F: \overline{E} \to Y$.

Pf) Given $f: E \to Y$, define $F: \overline{E} \to Y$ by $F(x) = \lim_{n\to\infty} f(x_n)$, where $x = \lim_{n\to\infty} x_n$ for some sequence $(x_n) \subseteq E$. Showing F is well-defined:

(i) Existence of (x_n): $\because x \in \overline{E}$.

(ii) $\lim_{n\to\infty} f(x_n)$ exists : (x_n) converges in $X \Rightarrow (x_n)$: Cauchy sequence

$\Rightarrow (f(x_n))$: Cauchy sequence ($\because$ f is uniformly continuous)

$\Rightarrow (f(x_n))$ converges .. ($\because$ Y is complete)

(iii) Independent of the choice of (x_n) : Suppose $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_n'$.

Consider $z_m := \begin{cases} x_{\frac{m+1}{2}}, & 2 \nmid m \\ x'_{\frac{m}{2}}, & 2 \mid m \end{cases}$, which is also a Cauchy sequence in X.

$\therefore \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} f(x_n')$ ($\because$ $(x_n), (x_n')$ are subsequences of (z_n))

$\therefore$ F is well-defined.

Showing F extends f: for any $x \in E$, take $x_n \equiv x$ for any $n \in \mathbb{N}$.

$\therefore$ $F(x) = \lim\limits_{n \to \infty} f(x) = f(x)$.

Showing F is uniformly continuous: Given $\varepsilon > 0$, by uniform continuity of f,

there exists $\tilde{\delta} > 0$ such that for any $x, x' \in E$ with $d(x, x') < \tilde{\delta}$, then $\rho(f(x), f(x')) < \frac{\varepsilon}{2}$

Choose $\delta := \frac{\tilde{\delta}}{3} > 0$; given any $z, z' \in \bar{E}$ with $d(z, z') < \delta$,

let $z = \lim\limits_{n \to \infty} x_n$ and $z' = \lim\limits_{m \to \infty} x'_m$ for some $(x_n), (x'_m) \subseteq E$.

there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $\begin{cases} d(z, x_n) < \delta \\ d(z', x'_m) < \delta \end{cases}$

$\therefore$ $d(x_n, x'_m) \leq d(x_n, z) + d(z, z') + d(z', x'_m) < 3\delta = \tilde{\delta}$

$\therefore$ $\rho(f(x_n), f(x'_m)) < \frac{\varepsilon}{2}$. Taking $m, n \to \infty$: $\rho(F(x), F(x')) \leq \frac{\varepsilon}{2} < \varepsilon$.

$\therefore$ F is uniformly continuous.

Q2) (HW 7, Q10) Show that $\begin{cases} x+y^4=0 \\ y-x^2=0.015 \end{cases}$ is solvable near $(x,y)=0\in(\mathbb{R}^2,\|\cdot\|)$.

Sol) Define $\Phi:\mathbb{R}^2\to\mathbb{R}^2$ by $\Phi=I+\Psi$, where $\Psi(x,y)=(\Psi_1(x,y),\Psi_2(x,y))=(y^4,-x^2)$.

Applying Perturbation of Identity to Φ: need to construct $r>0$ such that

$\Psi|_{\overline{B_r(0)}}:(\overline{B_r(0)},\|\cdot\|_2)\to(\mathbb{R}^2,\|\cdot\|_2)$ is a contraction. By [Lecture Note 3, P.8],

for any $(x_1,y_1),(x_2,y_2)\in\overline{B_r(0)}$, $\|\Psi(x_1,y_1)-\Psi(x_2,y_2)\|_2\le M\,\|(x_1,y_1)-(x_2,y_2)\|_2$, where

$$M=\sup_{\|(x,y)\|_2\le r}\left(\left(\frac{\partial\Psi_1}{\partial x}\right)^2+\left(\frac{\partial\Psi_1}{\partial y}\right)^2+\left(\frac{\partial\Psi_2}{\partial x}\right)^2+\left(\frac{\partial\Psi_2}{\partial y}\right)^2\right)^{\frac{1}{2}}=\sup_{\|(x,y)\|_2\le r}\left(0+(4y^3)^2+(-2x)^2+0\right)^{\frac{1}{2}}\le 2r\sqrt{4r^4+1}$$

Choose $r=\frac{1}{4}$: $M\le 2\cdot\frac{1}{4}\cdot\sqrt{\frac{1}{64}+1}=\frac{\sqrt{65}}{16}<1$. Hence $\Psi|_{\overline{B_{\frac14}(0)}}$ is a contraction.

By Perturbation of Identity, $\Phi(x)=y$ is solvable for any $y\in\overline{B_R(\Phi(0))}$, where

$R=(1-M)r>\frac{16-\sqrt{65}}{16}\cdot\frac{1}{4}>\frac{1}{16}$. In particular, $(0,0.015)\in\overline{B_R(0)}$.

$\therefore$ $\begin{cases} x+y^4=0 \\ y-x^2=0.015 \end{cases}$ is solvable for $\|(x,y)\|_2\le\frac{1}{4}$.